The problem solved here uses the analysis and information from the document called Second_Order_Linear_Equations.pdf, available on my web site, http://uhaweb.hartford.edu/ltownsend/.

Without damping (revised table to match the rest of the problem):

DG p. 181	m	k	x0	v0
1	10 kg	1000 N/m	1 m	0
2	10 kg	10 N/m	0	1 m/s
3	**10 kg**	**10 N/m**	**3 m**	**4 m/s**
4	1 kg	16 N/m	4 m	0
5	1 kg	25 N/m	0	3 m/s
6	9 kg	1 N/m	4 m	1 m/s

I will select problem 3 of the table above, which now matches those below. I choose this one as both initial conditions are non-zero. I will use this case throughout.

The differential equation is:

$$m\frac{d^2x}{dt^2}+kx=0 \qquad \text{or, equivalently,} \qquad \frac{d^2x}{dt^2}+\omega_0^2x=0 \text{ with } \omega_0^2=\frac{k}{m}$$

Problem 3 data:

$$m=10 \qquad k=10$$

Find ω_0:
$$\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{10}{10}}=1 \text{ rad/sec}$$

The solution is:
$$x(t)=C_1\cos t+C_2\sin t$$

Apply the initial conditions:

$$x(0)=x_0=3=C_1\cos 0+C_2\sin 0=C_1 \qquad \text{hence} \qquad C_1=3$$

To find C_2 we need the velocity, which is found from $v(t)=\frac{dx}{dt}$

$$v(t)=\frac{dx}{dt}=-C_1\sin t+C_2\omega_0\cos t$$

But we know $v(0) = v_0 = 4$ and $C_1 = 3$. So, $4 = -3\sin 0 + C_2 \cos 0$. i,e, $C_2 = 4$. The equation of motion is then

$$x(t) = 3\cos t + 4\sin t$$

With damping, $c = 10\sqrt{5} = 22.36$ kg/s**:**

The differential equation is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \qquad \frac{d^2x}{dt^2} + 2\xi\omega_0\frac{dx}{dt} + \omega_0^2 x = 0 \qquad 2\xi\omega_0 = \frac{c}{m}$$

with

Damping Ratio: $\xi = \dfrac{c}{2m\omega_0} = \dfrac{c}{c_0}$

Characteristic equation: $\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0$

Solution: $\lambda = \dfrac{-2\xi\omega_0 \pm \sqrt{(2\xi\omega_0)^2 - 4\omega_0^2}}{2}$ i.e. $\lambda = -\omega_0\left(\xi \pm \sqrt{\xi^2 - 1}\right)$

Discriminant: $D = \xi^2 - 1$

Problem 3 data:

$m = 10$ $k = 10$ $c = 10\sqrt{5} = 22.36$

First find ω_0, then ξ. Using an Excel spreadsheet with the above formulas gives the following information.

m	c	k	ω_0	$c_0 = 2m\omega_0$	$\xi = c/c_0$	$D = \xi^2 - 1$
10	$10\sqrt{5} = 22.36$	10	1	20	1.118	0.25>0

Given *D*, we see the case number on page 84 of your Schaum's Outline is Case 1. Hence

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

We find the two values of λ from

$$\lambda = -\omega_0\left(\xi \pm \sqrt{\xi^2 - 1}\right) = -(1)\left(1.118 \pm \sqrt{0.25}\right) = -1.118 \pm 0.5$$

Hence

$$\lambda == -0.618, -1.618$$

The solution is then

$$x(t) = C_1 e^{-0.618t} + C_2 e^{-1.618t}$$

We now apply initial conditions:

$x(0) = x_0 = 3$ and $v(0) = v_0 = 4$

We will need the velocity, $v(t) = \dfrac{dx}{dt}$

$$v(t) = -0.618 C_1 e^{-0.618t} - 1.618 C_2 e^{-1.618t}$$

Let $t = 0$. Then, knowing that $e^0 = 1$, we have

$$3 = C_1 + C_2$$
$$4 = -0.618 C_1 - 1.618 C_2$$

We now have two equations in two unknowns. I usually just use solver in the F2 menu:

Solve($3 = x + y$ and $4 = -0.618x - 1.618y$,{x,y})

The *and* is found in the catalog.

The result is $C_1 = x = 8.854$ and $C_2 = y = -5.854$. The equation of motion is then

$$x(t) = 8.854 e^{-0.618t} - 5.854 C_2 e^{-1.618t}$$

Now, due to all the numbers flying around, I will check my work.

$x(t) = 8.854 e^{-0.618t} - 5.854 C_2 e^{-1.618t}$ $\qquad x(0) = 8.854 - 5.854 = 3$

$v(t) = -5.472 e^{-0.618t} + 9.472 e^{-1.618t}$ $\qquad v(0) = -5.472 + 9.472 = 4$

The solution checks out. It also checks with deSolve.

deSolve(10x ' '+10$\sqrt{5}$ x '+10x=0 and x(0)=3 and x'(0)=4,t, x)

Note that by keeping the $\sqrt{5}$, the TI gives and answer with understandable exponents. Then, when you do ◊ ENTER, you see the coefficients match the above.

With a forcing function, $F(t) = e^{-2t}$ N:

The differential equation is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \qquad \text{or} \qquad \frac{d^2x}{dt^2} + 2\xi\omega_0\frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m}$$

with

Problem 3 data:

$m = 10 \qquad k = 10 \qquad c = 10\sqrt{5} = 22.36 \qquad F(t) = e^{-2t}$

Earlier we found that the two solutions of the homogeneous equation were

$x_1(t) = e^{-0.618t}$ and $x_2(t) = e^{-1.618t}$

The general solution to the differential equation is written

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + x_p(t)$$

where $x_p(t)$ is the solution of the nonhomogeneous equation

$$\frac{d^2 x_p}{dt^2} + 2\xi\omega_0 \frac{dx_p}{dt} + \omega_0^2 x_p = \frac{F(t)}{m}$$

with $x_1(t)$ and $x_2(t)$ being solutions of the homogeneous equation.

$$\frac{d^2 x_1}{dt^2} + 2\xi\omega_0 \frac{dx_1}{dt} + \omega_0^2 x_1 = 0$$

$$\frac{d^2 x_2}{dt^2} + 2\xi\omega_0 \frac{dx_2}{dt} + \omega_0^2 x_1 = 0$$

We assume that the particular solution has the form

$$x_p(t) = u_1(t) x_1(t) + u_2(t) x_2(t)$$

Inserting our two solutions we have

$$x_p(t) = u_1(t) e^{-0.618t} + u_2(t) e^{-1.618t}$$

To find $u_1(t)$ and $u_2(t)$ we use

$$\frac{du_1}{dx} = \frac{-F(t) x_2(t)}{mW(t)} \qquad \frac{du_2}{dx} = \frac{F(t) x_1(t)}{mW(t)}$$

where $W(t) = \begin{vmatrix} x_1 & x_2 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{vmatrix} = x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt}$.

For our problem,

$$W(t) = -1.618 e^{-0.618t} e^{-1.618t} + 0.618 e^{-0.618t} e^{-1.618t} = -e^{-0.618t} e^{-1.618t}$$

I left the two exponential terms separated to make the next step easier.

Then

$$\frac{du_1}{dx} = \frac{-e^{-2t} e^{-1.618t}}{-10 e^{-0.618t} e^{-1.618t}} = 0.1 e^{-1.382t}$$

$$\frac{du_2}{dx} = \frac{e^{-2t}e^{-0.618t}}{-10e^{-0.618t}e^{-1.618t}} = -0.1e^{-0.382t}$$

Performing the integrals, we find

$$u_1(t) = \int e^{-1.382t}\,dt = -\frac{e^{-1.382t}}{13.82}$$
$$u_2(t) = -\int e^{-0.382t}\,dt = \frac{e^{-0.382t}}{3.82}$$

The particular solution is therefore

$$x_p(t) = e^{-0.618t}\left\{-\frac{e^{-1.382t}}{13.82}\right\} + e^{-1.618t}\left\{\frac{e^{-0.382t}}{3.82}\right\} = -\frac{e^{-2t}}{13.82} + \frac{e^{-2t}}{3.82} = 0.189e^{-2t}$$

We write the full solution as

$$x(t) = C_1x_1(t) + C_2x_2(t) + x_p(t)$$

Plug in the functions

$$x(t) = C_1e^{-0.618t} + C_2e^{-1.618t} + 0.1894e^{-2t}$$

We now apply initial conditions.

$x(0) = x_0 = 3$ and $v(0) = v_0 = 4$

We will need the velocity, $v(t) = \frac{dx}{dt}$

$$v(t) = -0.618C_1e^{-0.618t} - 1.618C_2e^{-1.618t} - 0.3788e^{-2t}$$

At $t = 0$ we have

$$3 = C_1 + C_2 + 1.894$$
$$4 = -0.618C_1 - 1.618C_2 - 0.3788$$

We now have two equations in two unknowns. I usually just use solver in the F2 menu:

Solve($3 = x + y + 0.1894$ and $4 = -0.618x - 1.618y - 0.3788$,{x,y})

The result is $C_1 = x = 8.926$ and $C_2 = y = -6.116$. The equation of motion is then

$$x(t) = 8.926e^{-0.618t} - 6.116e^{-1.618t} + 0.1894e^{-2t}$$

Now, due to all the numbers flying around again, I will check my work.

$x(t) = 8.926e^{-0.618t} - 6.116e^{-1.618t} + 0.1894e^{-2t}$ $\qquad x(0) = 8.926 - 6.116 + 0.1894 = 2.999$

$v(t) = -5.516e^{-0.618t} + 9.896e^{-1.618t} - 0.3788e^{-2t} \qquad v(0) = -5.516 + 9.896 - 0.3788 = 4.001$

The solution checks out to within the appropriate accuracy.

DEsolve gets $x(t) = 8.926e^{-0.618t} - 6.116e^{-1.618t} - 0.1894e^{-2t}$

Note that the driven and damped solutions both decay to zero so there is no long time behavior.

For an example where the driving force is sinusoidal see the handout on my website, http://uhaweb.hartford.edu/ltownsend/, entitled Nonhomogeneous_Equations_Resonance.pdf. The link is called Nonhomogeneous Differential Equations and Resonance.