MTH 352 Second Order Linear Equations Fall 2009

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These problems are taken from Derrick and Grossman Elementary Differential Equations. They are spring problems. Please watch all the videos under the category of Spring-Mass Systems at the URL http://www.math.armstrong.edu/faculty/hollis/DEmovies/
to get a visual understanding of the solutions to the equations.

Without damping:

$$m\frac{d^2x}{dt^2}+kx=0 \qquad \frac{d^2x}{dt^2}+\omega_0^2x=0 \qquad \omega_0^2=\frac{k}{m}$$

$$x(t)=C_1\cos\omega_0 t+C_2\sin\omega_0 t$$

$$v(t)=\frac{dx}{dt}=-C_1\omega_0\sin\omega_0 t+C_2\omega_0\cos\omega_0 t$$

DG p. 177	m	k	x0	v0
1	10 kg	1000 N/m	1 m	0
3	10 kg	10 N/m	3 m	4 m/s
5	1 kg	25 N/m	0	3 m/s

Note that once you have plugged in x0 and v0, you will typically end up with two equations in two unknowns. If you have forgotten how to solve them, please download the appropriate handout from my website to see several analytical and TI-89 ways to solve these equations for C_1 and C_2. http://uhaweb.hartford.edu/ltownsend/

With damping:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0 \qquad \frac{d^2x}{dt^2}+2\xi\omega_0\frac{dx}{dt}+\omega_0^2x=0 \qquad 2\xi\omega_0=\frac{c}{m}$$

Damping Ratio: $\xi=\dfrac{c}{2m\omega_0}=\dfrac{c}{c_0}$

Characteristic equation: $\lambda^2+2\xi\omega_0\lambda+\omega_0^2=0$

Solution: $\lambda=\dfrac{-2\xi\omega_0\pm\sqrt{(2\xi\omega_0)^2-4\omega_0^2}}{2}$ i.e. $\lambda=-\omega_0\left(\xi\pm\sqrt{\xi^2-1^2}\right)$

Discriminant: $D=\xi^2-1^2$

DG p. 181	m	c	k	x0	v0
1	10 kg	200 kg/s	1000 N/m	1 m	0
2	10 kg	20 kg/s	10 N/m	0	1 m/s
3	10 kg	$10\sqrt{5}$ kg/s	10 N/m	3 m	4 m/s
4	1 kg	10 kg/s	16 N/m	4 m	0
5	1 kg	8 kg/s	25 N/m	0	3 m/s
6	9 kg	10 kg/s	1 N/m	4 m	1 m/s

First find ω_0, then ξ. I did all six using an Excel spreadsheet using the above formulas.

DG p. 181	ω_0	$c_0 = 2m\omega_0$	$\xi = c/c_0$	$D = \xi^2 - 1$
1	10	200	1	0
2	1	20	1	0
3	1	20	1.118	>0
4	4	8	1.25	>0
5	5	10	0.8	<0
6	0.3333	6	1.667	>0

Excel formulas:

◇	A	B	C	D	E	F	G	H	I	J
1	DG p. 181	m	c	k	x0	v0	omega 0	c0	squiggle	D
2	1	10	200	1000	1	0	=SQRT(D2/B2)	=2*B2*G2	=C2/H2	=0
3	2	10	20	10	0	1	=SQRT(D3/B3)	=2*B3*G3	=C3/H3	=0
4	3	10	=10*SQRT(5)	10	3	4	=SQRT(D4/B4)	=2*B4*G4	=C4/H4	>0
5	4	1	10	16	4	0	=SQRT(D5/B5)	=2*B5*G5	=C5/H5	>0
6	5	1	8	25	0	3	=SQRT(D6/B6)	=2*B6*G6	=C6/H6	<0
7	6	9	10	1	4	1	=SQRT(D7/B7)	=2*B7*G7	=C7/H7	>0

Given *D*, find the case number on page 84 of your Schaum's Outline with $\lambda = -\omega_0\left(\xi \pm \sqrt{\xi^2 - 1^2}\right)$ and use the appropriate formula to find $x(t)$ with C_1 and C_2.

Case 1	D>0	$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
Case 2	D<0	$x(t) = e^{at}\left(C_1 \cos bt + C_2 \sin bt\right)$ $\quad \lambda = a \pm ib$
Case 3	D=0	$x(t) = \left(C_1 + C_2 t\right)e^{\lambda t}$

Then find the velocity $v(t) = \dfrac{dx}{dt}$. Feel free to use your TI-89 for the derivative. Apply the initial conditions to find C_1 and C_2. Write the final version of displacement $x(t)$.

Note that you can check your answer using *deSolve* on the TI. For example, take problem 6 above. Use the original formulation as that is how the original data values are given.

$$9\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + x = 0 \qquad x(0) = x_0 = 4 \qquad \left.\frac{dx}{dt}\right|_{t=0} = v(0) = v_0 = 1$$

The TI command is

deSolve(9x''+10x'+x=0 and x(0)=4 and x'(0)=1,t, x)

and is in the catalog and the 2nd derivative is indicated by two entries of the apostrophe, ', not the quote, ")

With a forcing function:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F(t) \qquad \frac{d^2x}{dt^2}+2\xi\omega_0\frac{dx}{dt}+\omega_0^2x=\frac{F(t)}{m}$$

See Schaum's Outline chapters 11 (the method of undetermined coefficients) and 12 (variation of parameters). Chapter 11 requires an educated guess as well as paying attention to the solutions of the homogeneous equation. Chapter 12 is a plug and chug recipe and that is the chapter we will study. Here is the method.

1) Solve the homogeneous equation ($\frac{d^2x}{dt^2}+2\xi\omega_0\frac{dx}{dt}+\omega_0^2x=0$). Name the two solutions are $x_1(t)$ and $x_2(t)$. It does not matter which one is which but once you have chosen, do not change their names as we use them below.

2) The full solution to the second order nonhomogeneous ($F(t)\neq 0$) linear differential equation is

$$x(t)=C_1x_1(t)+C_2x_2(t)+x_p(t)$$

where $x_p(t)$ is called the particular solution. Note that since all our coefficients are positive (m, c, k, R, L, C), $x_1(t)$ and $x_2(t)$ decay over time so are transient. The steady state solution is given by $x_p(t)$ with the exception of a possible transient term in $x_p(t)$.

3) Assume that the particular solution, $x_p(t)$, is a sum of the two homogeneous solutions.

$$x_p(t)=u_1(t)x_1(t)+u_2(t)x_2(t)$$

Note that the coefficients depend on time. We now find them.

4) Plug $x_p(t)=u_1(t)x_1(t)+u_2(t)x_2(t)$ into the differential equation

$$\frac{d^2x_p}{dt^2}+2\xi\omega_0\frac{dx_p}{dt}+\omega_0^2x_p=\frac{F(t)}{m}.$$

After a lot of manipulation we find

$$\frac{du_1}{dx}=\frac{-F(t)x_2(t)}{W(t)} \qquad \frac{du_2}{dx}=\frac{F(t)x_1(t)}{W(t)}$$

$W(t)$ is called the Wronskian and is found from the determinant $W(t)=\begin{vmatrix} x_1 & x_2 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{vmatrix}$.

5) Integrate $\frac{du_1}{dx}=\frac{-F(t)x_2(t)}{W(t)}$ and $\frac{du_2}{dx}=\frac{F(t)x_1(t)}{W(t)}$ to get $u_1(t)$ and $u_2(t)$. Do not include the arbitrary constant, *C*.

6) Plug in $u_1(t)$ and $u_2(t)$ to get the answer. $x(t) = C_1x_1(t) + C_2x_2(t) + x_p(t)$ where

$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$.

7) Apply initial conditions to find C_1 and C_2. Write the final version of displacement $x(t)$.

DG p. 181	m	c	k	x0	v0	F(t)
1	10 kg	200 kg/s	1000 N/m	1 m	0	$\cos(2t)$
2	10 kg	20 kg/s	10 N/m	0	1 m/s	$\sin(t)$
3	10 kg	$10\sqrt{5}$ kg/s	10 N/m	3 m	4 m/s	e^{-2t}
4	1 kg	10 kg/s	16 N/m	4 m	0	te^{-t}
5	1 kg	8 kg/s	25 N/m	0	3 m/s	7
6	9 kg	10 kg/s	1 N/m	4 m	1 m/s	$3x + 5x^2$

Identify the transient and steady state components of the solutions. Note that these problems contain the same homogeneous equations as above so you don't need to find them again. These six problems are designed for you to practice solving nonhomogeneous equations. Recall that all integrals and derivatives may be done on your calculator or at web sites, such as integrals.com.

Series RLC Circuits.

RLC circuits have the same equation as springs but with different components. Compare the following two equations

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

where $\frac{dQ}{dt} = i(t)$ is the current as a time derivative of the charge.

With a simple change of variable, all the equations for the spring turn into the equations for RLC circuits. Note that the capacitor uses the letter C so be careful when naming the arbitrary constants.

See my website for handouts on RLC circuits from a differential equations and Laplace Transform point of view.

http://uhaweb.hartford.edu/ltownsend/