Non Homogeneous Equations - Resonance

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Initial equation to solve: $$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F\left(t\right) \tag{1}$$

Divide by a: $$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{1}{m}F\left(t\right) \tag{2}$$

Recast as $$\frac{d^2x}{dt^2}+2\omega_0\xi\frac{dx}{dt}+\omega_0^2x=\frac{1}{m}F\left(t\right) \tag{3}$$

Where: $$\xi=\frac{1}{2\omega_0}\frac{c}{m} \quad \text{and} \quad \omega_0=\sqrt{\frac{k}{m}} \tag{4}$$

First solve the homogeneous equation to get the complementary solution (solutions to the homogeneous equation),

$$\frac{d^2x}{dt^2}+2\omega_0\xi\frac{dx}{dt}+\omega_0^2x=0 \tag{5}$$

Solution to homogeneous equation, the particular solution, x_c

Assume $$x=e^{\lambda t} \tag{6}$$

The characteristic equation is: $$\lambda^2+2\omega_0\xi\lambda+\omega_0^2x=0 \tag{7}$$

Leading to $$\lambda=-\omega_0\left(\xi\pm\sqrt{\xi^2-1}\right)=\lambda_1,\lambda_2$$

(8)

The complementary solution is therefore $$x=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t} \tag{9}$$

DO NOT evaluate C_1 and C_2 using the complementary solution. You evaluate them after finding the particular solution then creating the full solution to the nonhomogeneous equation.

Solution to nonhomogeneous equation, the particular solution, x_p

Assume $f\left(t\right)=F_0\sin\left(\omega t\right)$ (10)

$$\frac{d^2x_p}{dt^2}+2\omega_0\xi\frac{dx_p}{dt}+\omega_0^2x_p=\frac{F_0}{m}\sin\left(\omega t\right) \tag{11}$$

Method 1. Variation of Parameters

Assume: $$x_p\left(t\right)=u_1\left(t\right)x_1\left(t\right)+u_2\left(t\right)x_2\left(t\right) \tag{12}$$

Where $$x_1=e^{\lambda_1 t} \qquad x_2=e^{\lambda_2 t} \tag{13a,b}$$

From a previous handout

$$u_1'(t) = \frac{-f(t)x_2(t)}{W(t)} \qquad u_2'(t) = \frac{f(t)x_1(t)}{W(t)} \qquad \text{with} \quad f(t) = \frac{F_0}{m}\sin(\omega t) \tag{14}$$

Where the Wronskian is

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t) = e^{\lambda_1 t}\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}e^{\lambda_2 t} = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t} \tag{15}$$

Recall $\qquad \lambda_2 = -\omega_0\left(\xi - \sqrt{\xi^2 - 1}\right) \qquad \lambda_1 = -\omega_0\left(\xi + \sqrt{\xi^2 - 1}\right)$

Then $\qquad \lambda_2 - \lambda_1 = 2\omega_0\sqrt{\xi^2 - 1} \qquad \lambda_1 + \lambda_2 = -2\omega_0\xi \qquad (16)$

Equations (14) become

$$u_1'(t) = -\left(\frac{F_0}{m}\right)\left(\frac{\sin(\omega t)e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)e^{\lambda_1 t}e^{\lambda_2 t}}\right) = -\left(\frac{F_0}{m(\lambda_2 - \lambda_1)}\right)\left(\sin(\omega t)e^{-\lambda_1 t}\right)$$

$$u_2'(t) = \left(\frac{F_0}{m}\right)\left(\frac{\sin(\omega t)e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)e^{\lambda_1 t}e^{\lambda_2 t}}\right) = \left(\frac{F_0}{m(\lambda_2 - \lambda_1)}\right)\left(\sin(\omega t)e^{-\lambda_2 t}\right) \tag{17}$$

Integrate equations (17): $\qquad u_1(x) = \int u_1'(x)dx \qquad$ and $\qquad u_2(x) = \int u_2'(x)dx \qquad (18)$

For both cases we need $\qquad \displaystyle\int \sin(\omega t)e^{-\lambda t}dt = \frac{e^{-\lambda t}}{\lambda^2 + \omega^2}\left(-\lambda\sin\omega t + \omega\cos\omega t\right) \qquad (19)$

Put it all together

$$x_p(t) = \left(\frac{F_0}{m(\lambda_2 - \lambda_1)}\right)\left(-e^{-\lambda_1 t}\frac{(-\lambda_1\sin\omega t + \omega\cos\omega t)}{(\lambda_1{}^2 + \omega^2)}e^{\lambda_1 t} + e^{-\lambda_2 t}\frac{(-\lambda_2\sin\omega t + \omega\cos\omega t)}{(\lambda_2{}^2 + \omega^2)}e^{\lambda_2 t}\right)$$

or

$$x_p(t) = \left(\frac{F_0}{m(\lambda_2 - \lambda_1)}\right)\left(-\frac{(-\lambda_1\sin\omega t + \omega\cos\omega t)}{(\lambda_1{}^2 + \omega^2)} + \frac{(-\lambda_2\sin\omega t + \omega\cos\omega t)}{(\lambda_2{}^2 + \omega^2)}\right) \tag{20}$$

Using Derrick and Grossman [Elementary Differential Equations, 4th ed, 1997, pp 182-3]

$$x_p(t) = b_1\sin\omega t + b_1\cos\omega t = A\sin(\omega t + \phi) \tag{21}$$

Their solution is

$$A = \frac{\dfrac{F_0}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + \left(2\xi\dfrac{\omega}{\omega_0}\right)^2}} \qquad \tan\phi = 2\xi\frac{\left(\dfrac{\omega}{\omega_0}\right)}{\left(\dfrac{\omega}{\omega_0}\right)^2 - 1} \tag{22a,b}$$

Define the relative frequency as $\omega_r = \dfrac{\omega}{\omega_0}$. Then equations (22) become

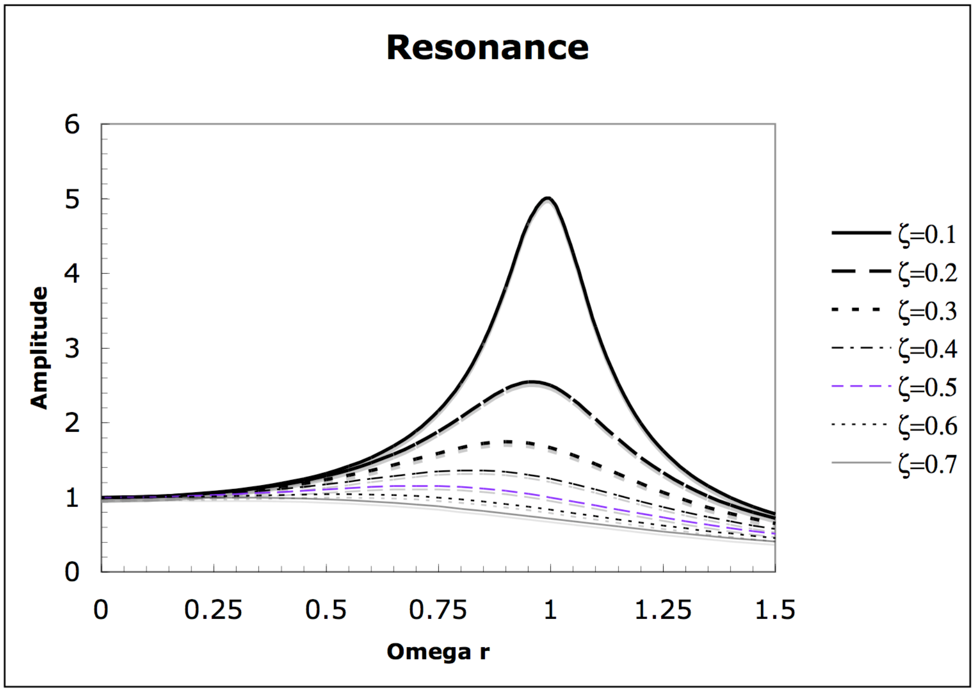
$$A = \frac{\frac{F_0}{k}}{\sqrt{\left[1-\omega_r^2\right]^2 + \left(2\xi\omega_r\right)^2}} \qquad \tan\phi = 2\xi\frac{\omega_r}{\omega_r^2 - 1} \tag{23a,b}$$

Note that the maximum value of this function is near $\omega_r = 1$. The actual value is found by taking the derivative of the amplitude then setting it equal to zero. The resulting location of the maximum is found to be

$$\omega_r = \sqrt{1-2\xi^2} \tag{24}$$

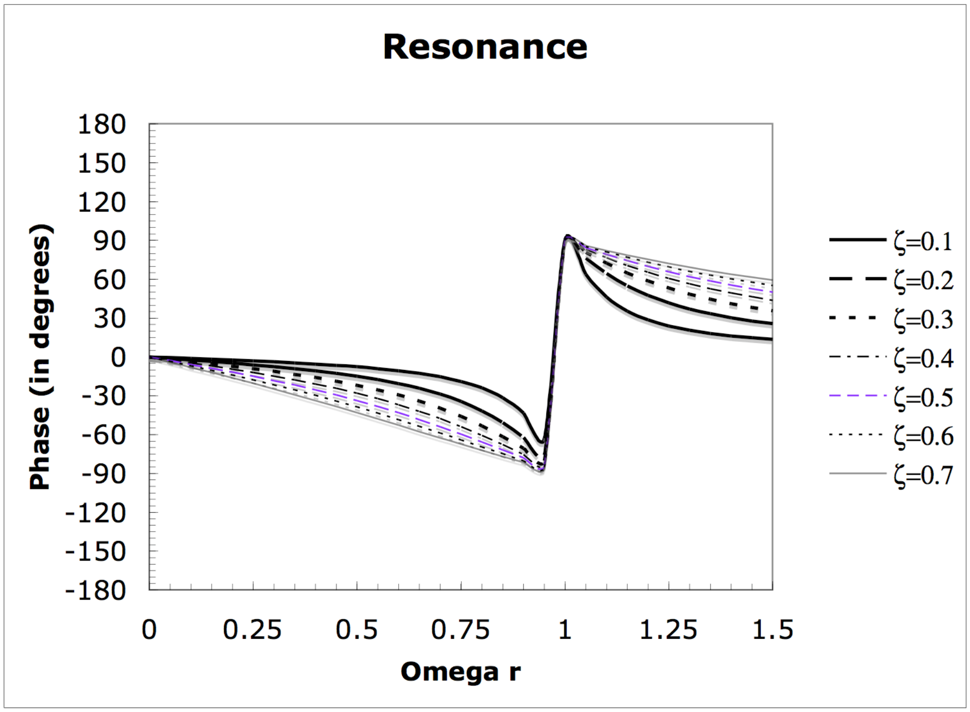
For this value of ω_r, the amplitude is found to be

$$A = \frac{\frac{F_0}{2k}}{\sqrt{\xi^2\left[1-\xi^2\right]}} \tag{25}$$



The value of $\frac{F_0}{k}$ in the above graph was taken to be unity: $\frac{F_0}{k} = 1$. The strength, F_0, of the applied force did not change, only its frequency. Hence, care must be taken when apply sinusoidal forces for small values of damping ratio, ζ. The less frictional loss there is to a system the more the resonance is fed. For the casae of no friction, the amplitude is infinite when $\omega_r = 1$, i.e. the system is being fed energy at its natural frequency and there is no loss of energy.

The phase values look like the typical arctan curves. When $\omega_r = 1$, the denominator vanishes so the angle is $\pm 90^0$.



Full solution to nonhomogeneous equation

The full solution to the problem is given by

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + x_p(t) \tag{26}$$

The solution has a transient part, the complementary solution $x_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, and the steady state part, $x_p(t)$. With no forcing function a damped harmonic oscillation will decay to no motion. The forcing function keeps the system oscillating by supplying energy to the system to counteract frictional losses.

RLC Circuits

The mechanical equation we are solving is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \sin\omega t \tag{27}$$

The series RLC circuit equation to solve is given by the Kirchhoff voltage law:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0 \sin\omega t \tag{28}$$

By appropriate identification of parameters, all the above analysis holds for RLC series circuits.

$$m \leftrightarrow L \qquad c \leftrightarrow R \qquad k \leftrightarrow \frac{1}{C} \qquad F_0 \leftrightarrow V_0 \tag{29}$$

Similar results hold for parallel RLC circuits using Kirchhoff's nodal analysis. See a previous handout for more information.