Model Refinement

Prof. Townsend Fall 2010

Growth and decay problems assume, initially, that the change in population in time, $\frac{d}{dt}N(t)$, is proportional to the current population, $N(t)$. The differential equation is therefore

$$\frac{d}{dt}N(t) = kN(t) \qquad \text{for growth}$$

$$\frac{d}{dt}N(t) = -kN(t) \qquad \text{for decay}$$

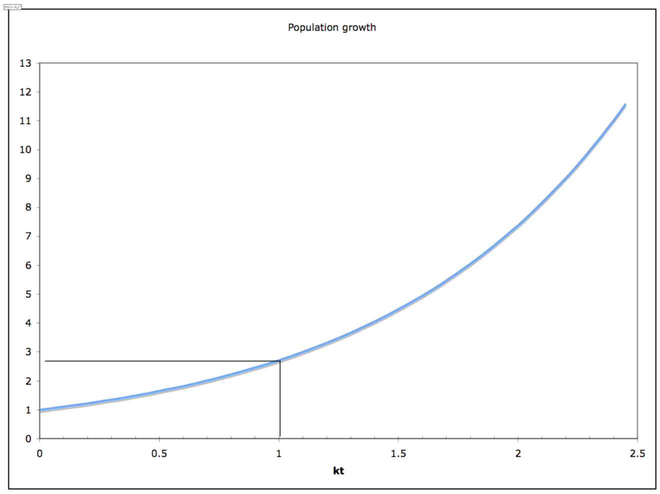
Life is easier if you always assume letters representing physical constants are positive entities, for example, the growth/decay constant, *k*. Put in the minus sign explicitly.

Since I want to talk about model refinement using the logistic equation (to be defined), I will consider the growth equation. The solution of $\frac{d}{dt}N(t) = kN(t)$ is

$$N(t) = N_0 e^{kt} \tag{1}$$

i.e. we start with a population of N_0 and it grows with time constant (1/*e* point) of $\tau = \frac{1}{k}$.

Below is a graph of $\frac{N(t)}{N_0}$ as a function of $\frac{t}{\tau}$. Note the use of scaled coordinates, a very important concept.



As you can see from the graph, the population grows without bound. This model clearly needs refinement since populations do not grow without bound.

Since example 7.7 in your book talks about mice, we will use mice as the population we are discussing. The model we described above only considers the birth rate and death rate.[1] It assumes that the population growth rate (birth rate minus death rate) per mouse is $\frac{1}{N}\frac{dN}{dt}$, a constant *k*.

$$\frac{1}{N}\frac{dN}{dt}=k$$

This equation is the starting model for mouse population. We now refine the model.

When there are too many mice, there starts to be competition for food and territory. This competition leads to an increased death rate that increases with the current population. Therefore we have to subtract a population dependent term from our initially constant growth rate. We assume a linear relationship between $\frac{1}{N}\frac{dN}{dt}$ and N.

$$\frac{1}{N}\frac{dN}{dt}=k-bN \tag{2}$$

This equation can be rewritten as

$$\frac{dN}{dt}=N\left(k-bN\right)$$

This equation is the logistic differential equation

Now we go about solving this equation. This equation is not linear but it is separable. Create the separable form of the equation

$$\frac{1}{N\left(k-bN\right)}dN=dt$$

In order to integrate the left hand side of the above equation, we perform the partial fraction expansion (which we will use extensively when using Laplace Transforms). We can see that the denominator on the left hand side (LHS) looks like a lowest common denominator (LCD). So, we find out what two fractions were combined.

$$\frac{1}{N\left(k-bN\right)}=\frac{A}{N}+\frac{B}{\left(k-bN\right)} \tag{3}$$

The constants A and B will be determined such that the LHS will equal the RHS.

Combine the two terms on the right hand side using their lowest common denominator

$$\frac{1}{N\left(k-bN\right)}=\frac{A\left(k-bN\right)+BN}{N\left(k-bN\right)}$$

The denominators are the same, so we equate the numerators.

$$1 = A\left(k - bN\right) + BN$$

$$1 = Ak + (B - bA)N$$

The above equation must be independent of *N* so $B = bA$. The remaining equation is $1 = Ak$. Solving for A and B we get

$$A = \frac{1}{k} \qquad B = \frac{b}{k}$$

The differential equation's LHS can now be written as

$$\frac{1}{N\left(k - bN\right)}dN = \frac{1}{k}\left(\frac{1}{N} + \frac{b}{\left(k - bN\right)}\right)dN = \frac{1}{k}\left(\frac{dN}{N} - \frac{dN}{\left(N - \frac{k}{b}\right)}\right)$$

so the equation to be integrated is

$$\frac{dN}{N} - \frac{dN}{\left(N - \frac{k}{b}\right)} = kdt \tag{4}$$

Integrating gives

$$\ln\left|N\right| - \ln\left|N - \frac{k}{b}\right| = kt + C$$

Using the division rule of logs gives

$$\ln\left|\frac{N}{N - \frac{k}{b}}\right| = kt + C$$

so

$$\frac{N}{N - \frac{k}{b}} = C'e^{kt}$$

Solving for *N* we get.

$$N = \left(\frac{b}{k}\right)\left(\frac{C'e^{kt}}{1 - C'e^{kt}}\right) \tag{5}$$

Now let's find C' by assuming the initial condition $N\left(0\right) = N_0$. The easiest way is to back up to the equation where *C*' is not buried inside of a complicated fraction.

$$\frac{N}{N-\frac{k}{b}}=C'e^{kt}$$

Letting *t*=0 we get

$$C'=\frac{N_0}{N_0-\frac{k}{b}} \tag{6}$$

Plugging this back into our solution then tidying gives

After tidying we get

$$N=N_0e^{kt}\left\{\frac{1}{1-N_0\left(\frac{b}{k}\right)\left(1-e^{kt}\right)}\right\} \tag{7}$$

I chose this particular way of displaying the equation so that if b=0, we recover our original solution.

If we are interested in what happens after a long time, we want to have the exponent be e^{-kt} since that form will die off with time. The above solution is therefore rewritten as

$$N=N_0\left\{\frac{1}{e^{-kt}\left[1-N_0\left(\frac{b}{k}\right)\right]+N_0\left(\frac{b}{k}\right)}\right\} \tag{8}$$

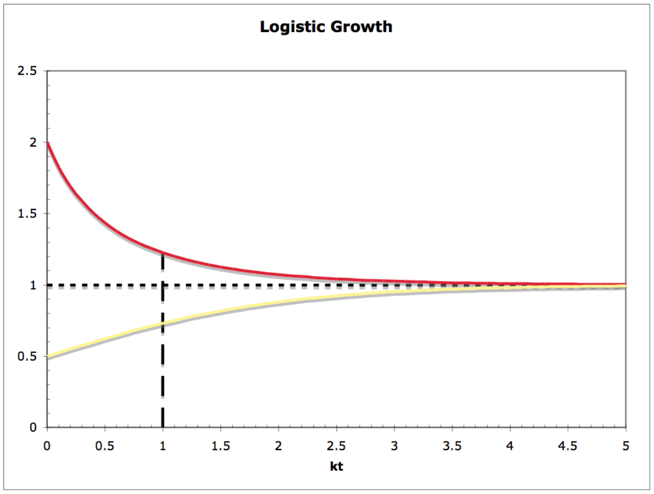
For long times the first term in the denominator disappears and the solution goes to

$$N\rightarrow\frac{k}{b}$$

Note that this equilibrium value is independent of initial conditions, only birth and death rates. If $N_0>\frac{k}{b}$, the population dies off. If $N_0<\frac{k}{b}$, the population grows until equilibrium is achieved. Name the equilibrium value $N_\infty=\frac{k}{b}$. Equation (8) becomes

$$\frac{N}{N_\infty}=\frac{N_0}{N_\infty}\left\{\frac{1}{\frac{N_0}{N_\infty}+e^{-kt}\left[1-\frac{N_0}{N_\infty}\right]}\right\} \tag{8}$$

To display the result graphically, I pick $N_0 = 2N_\infty$ then $N_0 = \frac{1}{2}N_\infty$. The two equations graphed are then $\frac{N}{N_0} = \frac{2}{2 - e^{-kt}}$ and $\frac{N}{N_0} = \frac{1}{1 + e^{-kt}}$



When a time span of five time constants has elapsed, both solutions have essentially reached equilibrium.

Note that I kept rewriting the solution depending on what time region I was considering. I also used scaled coordinates as much as possible so that the formalism is as useful as possible. In summary, if the differential equation is

$$\frac{dN}{dt} = N(k - bN)$$

Then the solution in scaled coordinates is the following

$$\frac{N}{N_\infty} = \frac{N_0}{N_\infty}\left\{\frac{1}{\frac{N_0}{N_\infty} + e^{-\frac{t}{\tau}}\left[1 - \frac{N_0}{N_\infty}\right]}\right\}$$

with $N_0 = N(0)$, $\tau = \frac{1}{k}$, and $N_\infty = \frac{k}{b} = \frac{1}{\tau b}$

Note that all parameters in the solution have physical significance. The solution clearly behaves at both small times and long times.

--------------------------------- Reference ---------------------------------

[1] Derrick and Grossman, "Elementary Differential Equations", 4th ed. (1997, Addison-Wesley), p 49.