Exam Scaling

Prof. Townsend | Fall 2007 (updated Fall 2011)

I look at two particular numbers to scale from one set of grades to another. Those numbers are the average value, also called the *mean*, and the *standard deviation* which is a measure of the spread in the data. Refer to your MTH 112/122 book for details of elementary statistics.

Let the grades be represented by x_i where the letter i stands for student number.
The average grade for N students is then

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where the bar over the x means *average value*. The symbol $\sum_{i=1}^{N} x_i$ means add up all N values for x_i.

The standard deviation, σ, is given by

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N-1}\left(x_i - \bar{x}\right)^2$$

Note that in what follows it does not matter whether $N-1$ or just N is used. The derivation follows for using either a sample or the full population.

Let me scale your grades for exam 2, x_i, such that they have the same $\bar{x}$ and σ as exam 1.
Your new grade, y_m, is written as

$$y_m = a x_{2_m} + b$$

Let $\bar{x}_1$ and σ_1 be those for exam 1. There were N_1 exam scores. I want exam 2 with N_2 exam scores to have the same statistics as those for exam 1. In other words, set

$$\bar{y} = \bar{x}_1 \qquad \text{and} \qquad \sigma_y = \sigma_1$$

These two equivalencies provide two equations for two unknowns, *a* and *b*. The resulting scaled grade y_m is

$$y_m = \frac{\sigma_1}{\sigma_2}\left(x_{2_m} - \bar{x}_2\right) + \bar{x}_1$$

Derivation

	Exam 1	Exam 2	Scaled Exam 2
Average Value	$\bar{x}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} x_{1_i}$	$\bar{x}_2 = \frac{1}{N_2}\sum_{k=1}^{N_2} x_{2_k}$	$\bar{y} = \frac{1}{N_2}\sum_{m=1}^{N_2} y_m$
Standard Deviation	$\sigma_1^2 = \frac{1}{N_1 - 1}\sum_{i=1}^{N_1-1}\left(x_{1_i} - \bar{x}_1\right)^2$	$\sigma_2^2 = \frac{1}{N_2 - 1}\sum_{k=1}^{N_2-1}\left(x_{2_k} - \bar{x}_2\right)^2$	$\sigma_y^2 = \frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(y_m - \bar{y}\right)^2$

Problem:

Make exam 2 have the same statistics as exam 1

$\bar{y} = \bar{x}_1$ and $\sigma_y = \sigma_1$ where $y_m = ax_{2_m} + b$

Solution:

Average value:

Plug $y_m = ax_{2_m} + b$ into $\bar{y} = \frac{1}{N_2}\sum_{m=1}^{N_2} y_m$

$$\bar{y} = \frac{1}{N_2}\sum_{m=1}^{N_2}\left(ax_{2_m} + b\right) = a\left(\frac{1}{N_2}\sum_{m=1}^{N_2} x_{2_m}\right) + b$$

$$\bar{y} = a\bar{x}_2 + b$$

Standard deviation:

Plug in $y_m = ax_{2_m} + b$ into $\sigma_y^2 = \frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(y_m - \bar{y}\right)^2$

$$\sigma_y^2 = \frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(ax_{2_m} + b - \bar{y}\right)^2$$

Add and subtract $a\bar{x}_2$ inside the summation to make the calculation easier.

$$\sigma_y^2 = \frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(a\left\{x_{2_m} - \bar{x}_2\right\} - \left\{\bar{y} - a\bar{x}_2 - b\right\}\right)^2$$

Recall from above that $\bar{y} = a\bar{x}_2 + b$. Therefore the second term vanishes and we have

$$\sigma_y^2 = \frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(a\left\{x_{2_m} - \bar{x}_2\right\}\right)^2 = a^2\left\{\frac{1}{N_2 - 1}\sum_{m=1}^{N_2-1}\left(x_{2_m} - \bar{x}_2\right)^2\right\}$$

$$\sigma_y^2 = a^2\sigma_2^2$$

Taking the square root of both sides we have

$$\sigma_y = a\sigma_2$$

Collect equations and solve for a and b:

$\overline{y} = a\overline{x}_2 + b$ and $\sigma_y = a\sigma_2$

Now set $\overline{y} = \overline{x}_1$ and $\sigma_y = \sigma_1$.

We therefore have

$\overline{x}_1 = a\overline{x}_2 + b$ and $\sigma_1 = a\sigma_2$

Solve for a and b.

$a = \frac{\sigma_1}{\sigma_2}$ and $b = \overline{x}_1 - a\overline{x}_2 = \overline{x}_1 - \frac{\sigma_1}{\sigma_2}\overline{x}_2$

The equation to be used for $y_m = ax_{2_m} + b$ is

$$y_m = \left(\frac{\sigma_1}{\sigma_2}\right)x_{2_m} + \left(\overline{x}_1 - \frac{\sigma_1}{\sigma_2}\overline{x}_2\right)$$

Final result:

$$y_m = \frac{\sigma_1}{\sigma_2}\left(x_{2_m} - \overline{x}_2\right) + \overline{x}_1$$

Q.E.D.