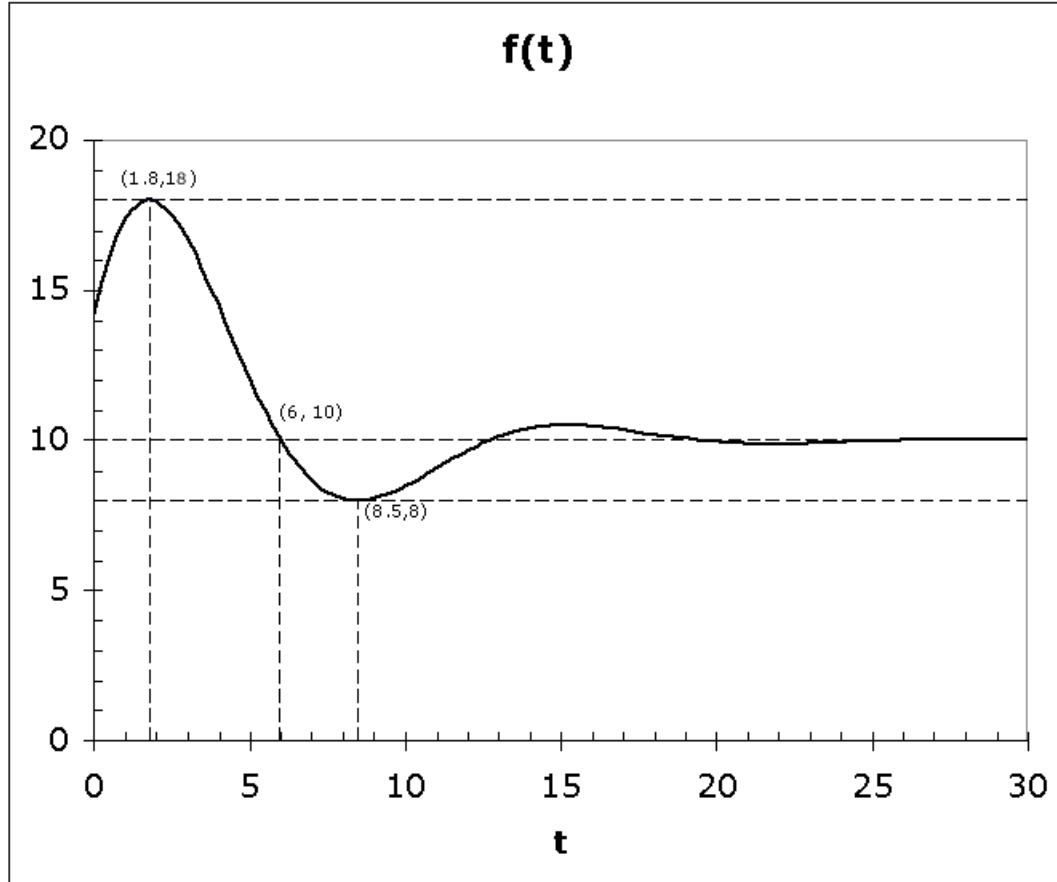


Analyzing Damped Oscillations

Problem (Meador, example 2-18, pp 44-48):

Determine the equation of the following graph.



The equation is assumed to be of the following form

$$f(t) = K_1 u(t) + K_2 e^{-\alpha t} \sin(\omega t + \theta) u(t) \quad (1)$$

where $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$ is the unit step function. (2)

The solution we will find is

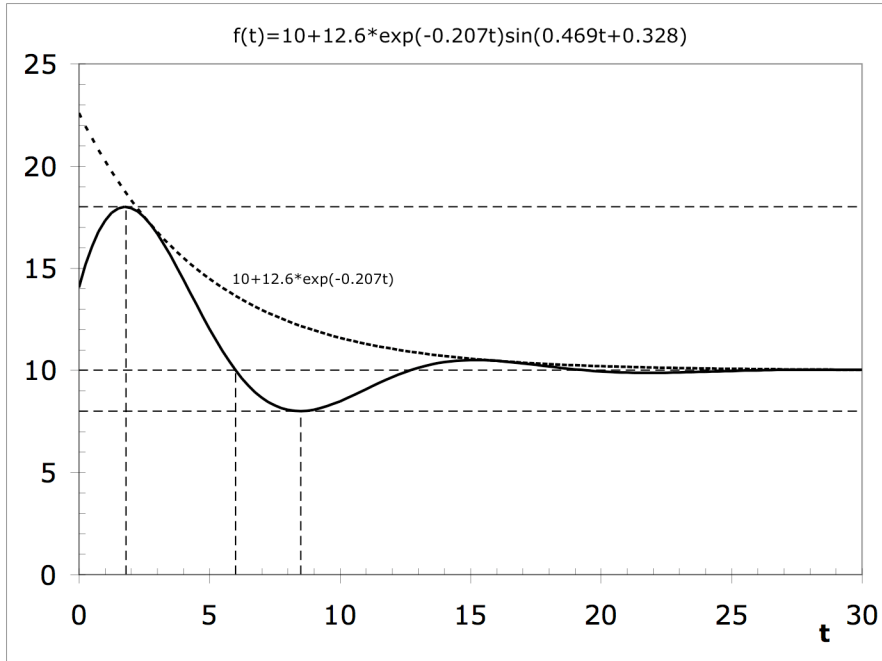
$$f(t) = 10u(t) + 12.6e^{-0.207t} \sin(0.469t + 0.328)u(t) \quad (3)$$

Solution:

We need to figure out the following parameters: K_1 , K_2 , α , ω , and θ .

Find K_1 :

We note that the second term goes to zero for large time due to the exponential decay of the envelope function, $f(t) = 10u(t) + 12.6e^{-0.207t}u(t)$.



We see that $\lim_{t \rightarrow \infty} f(t) = 10$ so $K_1 = 10$.

Find ω :

We note that the graph of $f(t)$ has maxima and minima. To find the values of the function at those points, take the derivative of $f(t) = K_1 + K_2e^{-\alpha t} \sin(\omega t + \theta)$ and set it equal to zero with $t > 0$.

$$\frac{d}{dt} f(t) = K_2 e^{-\alpha t} \{-\alpha \sin(\omega t + \theta) + \omega \cos(\omega t + \theta)\} = 0 \quad (4)$$

We rearrange to find that at the extrema

$$\tan(\omega t_{\max/\min} + \theta) = \frac{\omega}{\alpha} \quad (5)$$

Since the period of the tangent function is π , the phase angle between the first two extrema is π . Note the right hand side of equation (5) is constant. Therefore

$$8.5\omega + \theta = 1.8\omega + \theta + \pi \quad (6)$$

Solving for ω we get

$$\omega = \frac{\pi}{8.5 - 1.8} = 0.469 \text{ rad/sec} \quad (7)$$

Find θ :

To find the phase angle, θ , we look at the first "zero".

$$f(t) - K_1 = K_2 e^{-\alpha t} \sin(\omega t + \theta) = 0 \quad (8)$$

This value occurs when $\omega t + \theta = 0, \pi, 2\pi, \dots$. Since the sine is descending at $t=6$, we see

$$\omega * 6 + \theta = \pi \quad (9)$$

Since we now know $\omega = 0.469$, we get

$$\theta = \pi - 6 * 0.469 = 0.328 \text{ rad} \quad (10)$$

Find α :

We use the knowledge of t and $f(t)$ at $t=1.8$ and 8.5 .

$$f(1.8) - 10 = K_2 e^{-1.8\alpha} \sin(0.469 * 1.8 + 0.328)$$

$$f(8.5) - 10 = K_2 e^{-8.5\alpha} \sin(0.469 * 8.5 + 0.328)$$

or, equivalently,

$$8 = 0.922 K_2 e^{-1.8\alpha} \quad (11a)$$

$$-2 = -0.922 K_2 e^{-8.5\alpha} \quad (11b)$$

Divide these two equations to have the K_2 cancel each other thus we determine α .

$$4 = e^{6.7\alpha} \quad (12)$$

Solve for α .

$$\alpha = \frac{\ln 4}{6.7} = 0.207 \quad (13)$$

Find K_2 :

We now know all but K_2 . We find K_2 by evaluating the function at a known point.

$$18 = 10 + K_2 e^{-0.207 * 1.8} \sin(0.469 * 1.8 + 0.328) \quad (14)$$

We find $K_2 = 12.6$.

In summary, we find the equation of the original graph to be

$$f(t) = 10u(t) + 12.6e^{-2.07t} \sin(0.469t + 0.328)u(t) \quad (15)$$

A quandary

Finding α . When finding ω , we found

$$\tan(\omega t_{\max/\min} + \theta) = \frac{\omega}{\alpha}$$

Therefore,

$$\alpha = \frac{\omega}{\tan(\omega t_{\max/\min} + \theta)} = \frac{0.469}{\tan(0.469 * 1.8 + 0.328)} = 0.198 \quad (16)$$

However, this is not the same answer we got before which was 0.207. What's going on? It turns out that our graph reading is not quite right.

Look at some interesting points from the TI-89 from graphing equation (15)

t	f(t)
1.7636	18.0015 (maximum)
1.8	18.000148
5.9991	10.
8.462	8.000204 (minimum)
8.5	8.00058

The small number of significant figures in t is affecting the answer. Hence, you need to read the graph with greater accuracy, which may or may not be possible. However, this method is good enough to estimate parameters to within two significant figures, the accuracy of the time data.

The error in α is $\frac{0.207 - 0.198}{2} = 0.0045 \rightarrow 0.5\%$ which is certainly within the error built into the calculation.

Connection to the damped harmonic oscillator

For an undriven spring of spring constant k , mass m , and damping constant b , we have

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0 \quad (17)$$

Divide by m .

$$\frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \quad (18)$$

Now we transition to more informative parameters. Define

$$\omega_0^2 = \frac{k}{m} \quad \text{and} \quad 2\xi\omega_0 = \frac{c}{m} \quad (19)$$

Now equation (18) becomes

$$\frac{d^2x}{dt^2} + 2\xi\omega_0 \frac{dx}{dt} + \omega_0^2 x = 0 \quad (20)$$

Assume a solution of the form $x = Ce^{\lambda t}$ where C is a constant and λ is to be determined so that this solution is actually a solution of equation (20). When we plug it into equation (20), $Ce^{\lambda t}$ is in every term so factors out (it is never 0) and we get the following equation, known as the characteristic equation.

$$\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0 \quad (21)$$

The solution to this quadratic equation is

$$\lambda = -\omega_0 \left\{ \xi \pm \sqrt{\xi^2 - 1} \right\} \quad (22)$$

When $\xi < 1$ we get

$$\lambda = -\omega_0 \left\{ \xi \pm j\sqrt{1 - \xi^2} \right\} \quad (23)$$

We use Euler's relationship

$$e^{\pm jx} = \cos x \pm j \sin x \quad (24)$$

We therefore have the solution

$$x = e^{-\omega_0 \xi t} \left\{ a e^{j\omega t} + b e^{-j\omega t} \right\} \quad (25)$$

Other forms are (see Appendices A and B)

$$x = e^{-\omega_0 \xi t} \left\{ C_1 \cos(\omega t) + C_2 \sin(\omega t) \right\} \quad (26)$$

$$x = e^{-\omega_0 \xi t} A \sin(\omega t + \phi) \quad (27)$$

where

$$\omega = \omega_0 \sqrt{1 - \xi^2} \quad (28)$$

We now relate the harmonic oscillator to the original problem.

$$\omega = \omega_0 \sqrt{1 - \xi^2} = 0.469 \quad (29)$$

$$\alpha = \omega_0 \xi = 0.207 \quad (30)$$

As we did before, we divide the two equations

$$\frac{\omega}{\alpha} = \frac{\sqrt{1 - \xi^2}}{\xi} = \frac{0.469}{0.207} = 2.266 \quad (31)$$

Rewriting we get

$$\sqrt{1 - \xi^2} = 2.266\xi \quad (32)$$

Solve for ξ .

$$\xi = 0.404 \quad (33)$$

We now use equation (30) to solve for ω_0 .

$$\omega_0 = \frac{0.207}{\xi} = \frac{0.207}{0.404} = 0.512 \text{ rad/sec} \quad (34)$$

We can now back out of the above $\frac{c}{m}$ and $\frac{k}{m}$.

$$\frac{k}{m} = \omega_0^2 = 0.263 \quad (35)$$

$$\frac{c}{m} = 2\xi\omega_0 = 2(0.404)(0.512) = 0.414 \quad (36)$$

In summary, our system has the following parameter values:

Parameter	Value	Interrelationship
$\frac{k}{m}$	0.263	
$\frac{c}{m}$	0.414	
ω_0	0.512 rad/sec	$\omega_0 = \sqrt{\frac{k}{m}}$
ξ	0.404	$\xi = \frac{c}{2m\omega_0} = \frac{c}{2\sqrt{km}}$

Note that all these parameters involve ratios of hardware values, m , c , and k . The performance of the system depends on the scaled parameters, ω_0 and ξ , as shown in column 3 above.

Some questions arise with this system.

1) How do we get to critical damping?

That occurs when $\xi = 1$ so

$$c_{crit} = 2m\omega_0 = 2\sqrt{km} \quad (37)$$

The above relationship provides a way to determine the effect of changing at least one of the physical parameters to achieve critical damping. We need to choose m , c , and k so that equation (37) holds and we achieve critical damping. Or, we can use equation (37) to stay away from critical damping – make sure that m , c , and k are chosen such that our system is far from the satisfying equation (37).

2) If, instead, you wish to specify the period of oscillation, T .

You are then you are choosing m , c , and k such that equation (38) provides the desired value of T .

$$\omega = \omega_0 \sqrt{1 - \xi^2} = \sqrt{\omega_0^2 - (\omega_0 \xi)^2} = \sqrt{\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2} = \frac{2\pi}{T} \quad (38)$$

3) If you wish to specify the damping time constant, τ , then you are specifying

$$\xi \omega_0 = \frac{c}{2m} = \frac{1}{\tau} \quad (39)$$

We need to choose c and m so that equation (39) holds for the chosen value of τ .

So, analyzing the graph provides data on your system, which you can then change to your desired specifications.

Connection to RLC circuits

For a series RLC circuit the Kirchhoff Voltage Law (KVL) with no voltage source is

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t) = 0 \quad (40)$$

or

$$\frac{d^2 Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC} Q = E(t) = 0 \quad (41)$$

With the appropriate definitions,

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad \text{and} \quad \xi = \frac{R}{2L\omega_0} \quad (42)$$

the damped spring analysis applies.

Note that according to Pisano, the canonical form for electrical problems is

$$\frac{d^2 x}{dt^2} + \frac{\omega_n}{Q} \frac{dx}{dt} + \omega_n^2 x = 0 \quad (43)$$

Compare this equation with equation (20), the canonical form for mechanical systems.

$$\frac{d^2 x}{dt^2} + 2\xi\omega_0 \frac{dx}{dt} + \omega_0^2 x = 0 \quad (20)$$

Appendix A

Harmonic Identities

Derrick and Grossman page 130

Identity #1: $a \cos \omega t + b \sin \omega t = A \cos(\omega t - \delta)$ (1)

With $A = \sqrt{a^2 + b^2}$, $\cos \delta = \frac{a}{A}$, $\sin \delta = \frac{b}{A}$ hence $\tan \delta = \frac{b}{a}$

When determining δ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \delta = \frac{a}{A}$	$\sin \delta = \frac{b}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+
III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof:

Use $\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$
 So $A \cos(\omega t - \delta) = \cos \omega t (A \cos \delta) + \sin \omega t (A \sin \delta)$
 Compare with $A \cos(\omega t - \delta) = a \cos \omega t + b \sin \omega t$

Identity #2: $a \cos \omega t + b \sin \omega t = A \sin(\omega t + \phi)$ (2)

With $A = \sqrt{a^2 + b^2}$, $\cos \phi = \frac{b}{A}$, $\sin \phi = \frac{a}{A}$, hence $\tan \phi = \frac{a}{b}$

When determining δ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \phi = \frac{b}{A}$	$\sin \phi = \frac{a}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+

III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof:

Use $\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi$
 So $A \sin(\omega t + \phi) = \cos \omega t (A \sin \phi) + \sin \omega t (A \cos \phi)$
 Compare with $A \sin(\omega t + \phi) = a \cos \omega t + b \sin \omega t$

TI-89: *F2/Trig/tExpand* and *F2/Trig/tCollect* will apply the trig identity.

To find the phase angle, use vectors: $[A \cos \phi, A \sin \phi]$ Polar.

i.e. $[b,a]$ Polar with a similar result for identity 1. Note: $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$

Appendix B Euler's Identity

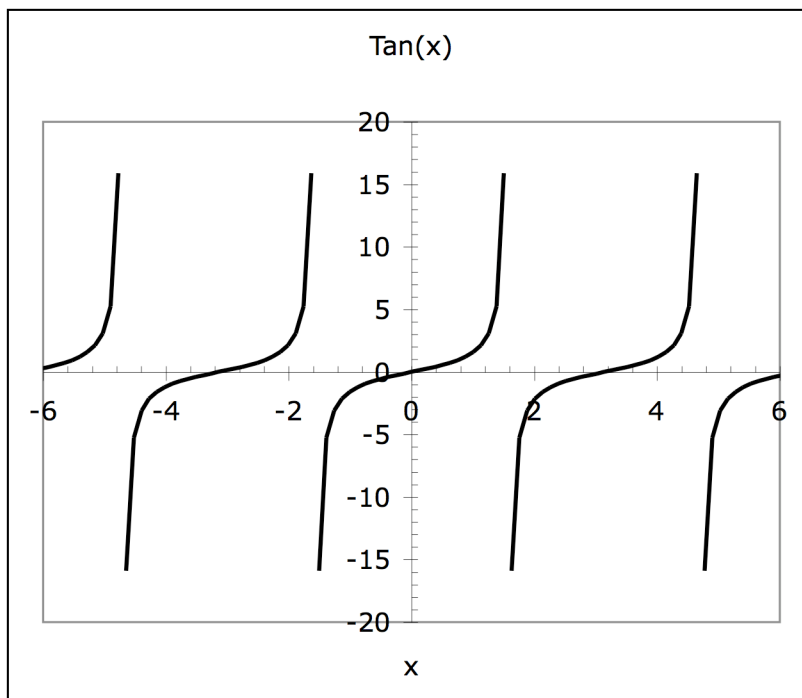
$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t), \quad e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

Therefore

$$ae^{j\omega t} + be^{-j\omega t} = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

Appendix C Period of the Tangent Function

From the graph below, you can see that the period of the tangent is π .



Appendix D

References

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